

Instantons in six dimensions and twistors

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Abstract

We consider homogeneous spaces $SU(4)/U(3)$ and $Sp(2)/Sp(1) \times U(1)$, both representing the complex projective space \mathbb{CP}^3 , as well as the natural twistor spaces $SU(4)/U(2) \times U(1)$ and $Sp(2)/U(1) \times U(1)$ fibred over the above cosets with \mathbb{CP}^2 and \mathbb{CP}^1 fibres, respectively. We describe the relation between Hermitian Yang-Mills connections (instantons) on complex vector bundles over \mathbb{CP}^3 and holomorphic bundles over the associated twistor spaces.

1 Introduction and summary

Let us consider an oriented real four-manifold X^4 with a Riemannian metric g and the principal bundle $P(X^4, SO(4))$ of orthonormal frames over X^4 . The (metric) twistor space $\text{Tw}(X^4)$ of X^4 can be defined as an associated bundle [1]

$$\text{Tw}(X^4) = P \times_{SO(4)} SO(4)/U(2) \quad (1.1)$$

with the canonical projection $\text{Tw}(X^4) \rightarrow X^4$. This space parametrizes the almost complex structures on X^4 compatible with the metric g (almost Hermitian structures). It was shown in [1, 2] that if the Weyl tensor of (X^4, g) is anti-self-dual then the almost complex structure on the twistor space $\text{Tw}(X^4)$ is integrable. Furthermore, it was proven that the rank r complex vector bundle E over X^4 with an anti-self-dual gauge potential A over such X^4 lifts to a holomorphic bundle \hat{E} over complex twistor space $\text{Tw}(X^4)$ [1, 3].

The essence of the canonical twistor approach is to establish a correspondence between four-dimensional space X^4 (or its complex version) and complex twistor space $\text{Tw}(X^4)$ of X^4 . Using this correspondence, one transfers data given on X^4 to data on $\text{Tw}(X^4)$ and vice versa. In twistor theory one considers *holomorphic* objects h on $\text{Tw}(X^4)$ (Čech cohomology classes, holomorphic vector bundles etc.) and transforms them to objects f on X^4 which are constrained by some differential equations [1]-[4]. Thus, the main idea of twistor theory is to encode solutions of some differential equations on X^4 in holomorphic data on the complex twistor space $\text{Tw}(X^4)$ of X^4 .

The twistor approach was recently extended to maximally supersymmetric Yang-Mills theory on \mathbb{C}^6 [5]. It was also generalized to Abelian [6, 7] and non-Abelian [8] holomorphic principal 2-bundles over the twistor space $Q'_6 \subset \mathbb{C}P^7 \setminus \mathbb{C}P^3$, corresponding to self-dual Lie-algebra-valued 3-forms on \mathbb{C}^6 . These forms are the most important objects needed for constructing (2,0) superconformal field theories in six dimensions, which are believed to describe stacks of M5-branes in the low-energy limit of M-theory [9].

The papers [5]-[8] (see also references therein) show that the twistor methods can be useful in higher-dimensional Yang-Mills and superconformal field theories. However, there are some problems in generalizing the twistor approach to higher dimensions. Namely, let X^{2n} be a Riemannian manifold of dimension $2n$. The metric twistor space of X^{2n} is defined as the bundle $\text{Tw}(X^{2n}) \rightarrow X^{2n}$ of almost Hermitian structures on X^{2n} associated with the principal bundle of orthonormal frames of X^{2n} , i.e.

$$\text{Tw}(X^{2n}) := P(X^{2n}, SO(2n)) \times_{SO(2n)} SO(2n)/U(n) . \quad (1.2)$$

It is well known that $\text{Tw}(X^{2n})$ can be endowed with an almost complex structure \mathcal{J} , which is integrable if and only if the Weyl tensor of X^{2n} vanishes when $n > 2$ [10]. This is too strong a restriction. However, if the manifold X^{2n} has a G -structure (not necessarily integrable), then one can often find a subbundle \mathcal{Z} of $\text{Tw}(X^{2n})$ associated with the G -structure bundle $P(X^{2n}, G)$ for $G \subset SO(2n)$, such that an induced almost complex structure (also called \mathcal{J}) on \mathcal{Z} is integrable. Many examples were considered in [10]-[14]. Another problem is that, in higher dimensions, solutions of differential equations do not always correspond to holomorphic objects on the reduced twistor space \mathcal{Z} (even if \mathcal{Z} is a complex manifold). In [15] this was shown for the example of Yang-Mills instantons on the six-sphere S^6 , which has the reduced complex twistor space $\mathcal{Z} = G_2/U(2)$. For the definition of instanton equations in dimensions higher than four and for some instanton solutions see e.g. [16]-[23].

In this paper we discuss instantons in gauge theory on the complex projective space \mathbb{CP}^3 by using twistor theory. Natural instanton-type equations in six dimensions are the Donaldson-Uhlenbeck-Yau (DUY) equations [17], which are $SU(3)$ invariant but not invariant under the $SO(6)$ Lorentz-type rotations of orthonormal frames. Hence, for their description one should consider reduced twistor spaces. The DUY equations are well defined on six-dimensional Kähler manifolds M (as well as on nearly Kähler spaces [24, 25, 26]), and their solutions are natural connections \mathcal{A} on holomorphic vector bundles $\mathcal{E} \rightarrow M$ [17]. On the example of $M = \mathbb{CP}^3$ we will show that such bundles $(\mathcal{E}, \mathcal{A})$ are pulled back to holomorphic vector bundles $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}})$ over the reduced twistor space $\mathcal{Z} \subset \text{Tw}(\mathbb{CP}^3)$ trivial along the fibres of the fibration $\mathcal{Z} \rightarrow \mathbb{CP}^3$ with $\mathcal{Z} = SU(4)/U(2) \times U(1)$ or $\mathcal{Z} = Sp(2)/U(1) \times U(1)$, depending on the chosen holonomy group. Note that this correspondence, valid for the reduced twistor spaces $\mathcal{Z} \hookrightarrow \text{Tw}(\mathbb{CP}^3)$, does not hold for the metric twistor space $\text{Tw}(\mathbb{CP}^3)$.

2 Kähler and quasi-Kähler structure on \mathbb{CP}^3

Coset representation of S^4 . Let us consider the group $Sp(2)$ fibred over $S^4 = Sp(2)/Sp(1) \times Sp(1)$,

$$Sp(2) \rightarrow S^4 \quad (2.1)$$

i.e. consider $Sp(2)$ as the fibre bundle $P(S^4, Sp(1) \times Sp(1))$ with the structure group $Sp(1) \times Sp(1)$. Local sections of the fibrations (2.1) can be chosen as 4×4 matrices

$$Q := f^{-\frac{1}{2}} \begin{pmatrix} \mathbf{1}_2 & -x \\ x^\dagger & \mathbf{1}_2 \end{pmatrix} \quad \text{and} \quad Q^{-1} = Q^\dagger = f^{-\frac{1}{2}} \begin{pmatrix} \mathbf{1}_2 & x \\ -x^\dagger & \mathbf{1}_2 \end{pmatrix} \in Sp(2) \subset SU(4), \quad (2.2)$$

where

$$x = x^\mu \tau_\mu, \quad x^\dagger = x^\mu \tau_\mu^\dagger, \quad f := 1 + x^\dagger x = 1 + r^2 = 1 + \delta_{\mu\nu} x^\mu x^\nu, \quad (2.3)$$

and matrices

$$(\tau_\mu) = (-i\sigma_i, \mathbf{1}_2) \quad \text{and} \quad (\tau_\mu^\dagger) = (i\sigma_i, \mathbf{1}_2) \quad (2.4)$$

obey

$$\begin{aligned} \tau_\mu^\dagger \tau_\nu &= \delta_{\mu\nu} \cdot \mathbf{1}_2 + \eta_{\mu\nu}^\dagger i\sigma_i =: \delta_{\mu\nu} \cdot \mathbf{1}_2 + \eta_{\mu\nu}, \quad \{\eta_{\mu\nu}^\dagger\} = \{-\eta_{\nu\mu}^\dagger\} = \{\varepsilon_{jk}^\dagger, \mu=j, \nu=k; \delta_j^i, \mu=j, \nu=4\}, \\ \tau_\mu \tau_\nu^\dagger &= \delta_{\mu\nu} \cdot \mathbf{1}_2 + \bar{\eta}_{\mu\nu}^\dagger i\sigma_i =: \delta_{\mu\nu} \cdot \mathbf{1}_2 + \bar{\eta}_{\mu\nu}, \quad \{\bar{\eta}_{\mu\nu}^\dagger\} = \{-\bar{\eta}_{\nu\mu}^\dagger\} = \{\varepsilon_{jk}^\dagger, \mu=j, \nu=k; -\delta_j^i, \mu=j, \nu=4\}. \end{aligned} \quad (2.5)$$

Here $\{x^\mu\}$ are local coordinates on an open set $\mathcal{U} \subset S^4$. Matrices (2.2) are representative elements for the coset space $S^4 = Sp(2)/Sp(1) \times Sp(1)$.

Flat connection on S^4 . Consider a flat connection \mathcal{A}_0 on the trivial vector bundle $S^4 \times \mathbb{C}^4 \rightarrow S^4$ given by the one-form

$$\mathcal{A}_0 = Q^{-1} dQ =: \begin{pmatrix} A^- & -\phi \\ \phi^\dagger & A^+ \end{pmatrix}, \quad (2.6)$$

where from (2.2) we obtain

$$A^- = \frac{1}{f} \bar{\eta}_{\mu\nu} x^\mu dx^\nu =: \begin{pmatrix} \alpha_- & -\bar{\beta}_- \\ \beta_- & -\alpha_- \end{pmatrix} \in su(2), \quad (2.7)$$

$$A^+ = \frac{1}{f} \eta_{\mu\nu} x^\mu dx^\nu =: \begin{pmatrix} \alpha_+ & -\bar{\beta}_+ \\ \beta_+ & -\alpha_+ \end{pmatrix} \in su(2) , \quad (2.8)$$

$$\phi = \frac{1}{f} dx = -\frac{i}{f} \begin{pmatrix} dx^3 + i dx^4 & dx^1 - i dx^2 \\ dx^1 + i dx^2 & -(dx^3 - i dx^4) \end{pmatrix} = -\frac{i}{f} \begin{pmatrix} dz & d\bar{y} \\ dy & -d\bar{z} \end{pmatrix} =: \begin{pmatrix} \theta^2 & \theta^{\bar{1}} \\ -\theta^1 & \theta^{\bar{2}} \end{pmatrix} , \quad (2.9)$$

with

$$\alpha_+ = \frac{1}{2f} (\bar{y} dy + \bar{z} dz - y d\bar{y} - z d\bar{z}) , \quad \beta_+ = \frac{1}{f} (y dz - z dy) , \quad (2.10)$$

$$\alpha_- = \frac{1}{2f} (\bar{y} dy + z d\bar{z} - y d\bar{y} - \bar{z} dz) , \quad \beta_- = \frac{1}{f} (y d\bar{z} - \bar{z} dy) , \quad (2.11)$$

$$\theta^1 := \frac{idy}{1+r^2} , \quad \theta^2 := -\frac{idz}{1+r^2} \quad \text{and} \quad \theta^{\bar{1}} := -\frac{id\bar{y}}{1+r^2} , \quad \theta^{\bar{2}} := \frac{id\bar{z}}{1+r^2} . \quad (2.12)$$

Here, the bar denotes complex conjugation.

Coset representation of S^2 . Let us consider the Hopf bundle

$$S^3 \rightarrow S^2 \quad (2.13)$$

over the Riemann sphere $S^2 \cong \mathbb{CP}^1$ and the one-monopole connection a on the bundle (2.13) having in the local complex coordinate $\zeta \in \mathbb{CP}^1$ the form

$$a = \frac{1}{2(1 + \zeta\bar{\zeta})} (\bar{\zeta} d\zeta - \zeta d\bar{\zeta}) . \quad (2.14)$$

Consider a local section of the bundle (2.13) given by the matrix

$$g = \frac{1}{(1 + \zeta\bar{\zeta})^{\frac{1}{2}}} \begin{pmatrix} 1 & -\bar{\zeta} \\ \zeta & 1 \end{pmatrix} \in \text{SU}(2) \cong S^3 \quad (2.15)$$

and introduce the $su(2)$ -valued one-form (flat connection)

$$g^{-1} dg =: \begin{pmatrix} a & -\theta^{\bar{3}} \\ \theta^3 & -a \end{pmatrix} \quad (2.16)$$

on the bundle $S^2 \times \mathbb{C}^2 \rightarrow S^2$, where

$$\theta^3 = \frac{d\zeta}{1 + \zeta\bar{\zeta}} \quad \text{and} \quad \theta^{\bar{3}} = \frac{d\bar{\zeta}}{1 + \zeta\bar{\zeta}} \quad (2.17)$$

are the forms of type (1,0) and (0,1) on \mathbb{CP}^1 and a is the one-monopole gauge potential (2.14).

Twistor space $\text{Tw}(S^4)$. Let us introduce 4×4 matrices

$$G = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & g \end{pmatrix} \quad \text{and} \quad \hat{Q} = QG \in \text{Sp}(2) \subset \text{SU}(4) , \quad (2.18)$$

where Q and g are given in (2.2) and (2.15). The matrix \hat{Q} is a local section of the bundle

$$\text{Sp}(2) \rightarrow \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) =: \mathcal{M} . \quad (2.19)$$

Let us consider a trivial complex vector bundle $\mathcal{M} \times \mathbb{C}^4 \rightarrow \mathcal{M}$ with the flat connection

$$\hat{\mathcal{A}}_0 = \hat{Q}^{-1} d\hat{Q} = G^{-1} \mathcal{A}_0 G + G^{-1} dG =: \begin{pmatrix} \hat{A}^- & -\hat{\phi} \\ \hat{\phi}^\dagger & \hat{A}^+ \end{pmatrix}, \quad (2.20)$$

where

$$\hat{\phi} = \phi g =: \begin{pmatrix} \hat{\theta}^2 & \hat{\theta}^{\bar{1}} \\ -\hat{\theta}^1 & \hat{\theta}^{\bar{2}} \end{pmatrix}, \quad \hat{A}^- = A^- = \begin{pmatrix} \alpha_- & -\bar{\beta}_- \\ \beta_- & -\alpha_- \end{pmatrix}, \quad \hat{A}^+ =: \begin{pmatrix} \hat{\alpha}_+ & -\hat{\theta}^{\bar{3}} \\ \hat{\theta}^3 & -\hat{\alpha}_+ \end{pmatrix}, \quad (2.21)$$

with α_- , β_- given in (2.11) and

$$\hat{\alpha}_+ := \frac{1}{1 + \zeta \bar{\zeta}} \left\{ (1 - \zeta \bar{\zeta}) \alpha_+ + \bar{\zeta} \beta_+ - \zeta \bar{\beta}_+ + \frac{1}{2} (\bar{\zeta} d\zeta - \zeta d\bar{\zeta}) \right\}, \quad (2.22)$$

$$\hat{\theta}^1 := \frac{1}{(1 + \zeta \bar{\zeta})^{\frac{1}{2}}} (\theta^1 - \zeta \theta^{\bar{2}}), \quad \hat{\theta}^2 := \frac{1}{(1 + \zeta \bar{\zeta})^{\frac{1}{2}}} (\theta^2 + \zeta \theta^{\bar{1}}), \quad (2.23)$$

$$\hat{\theta}^3 := \frac{1}{(1 + \zeta \bar{\zeta})} (d\zeta + \beta_+ - 2\zeta \alpha_+ + \zeta^2 \bar{\beta}_+). \quad (2.24)$$

From flatness of the connection (2.20), $d\hat{\mathcal{A}}_0 + \hat{\mathcal{A}}_0 \wedge \hat{\mathcal{A}}_0 = 0$, we obtain the equations

$$d \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_+ - \alpha_- & \beta_- & -\frac{1}{2R} \hat{\theta}^{\bar{2}} \\ -\bar{\beta}_- & -\hat{\alpha}_+ + \alpha_- & \frac{1}{2R} \hat{\theta}^{\bar{1}} \\ \frac{R}{2\Lambda^2} \hat{\theta}^2 & -\frac{R}{2\Lambda^2} \hat{\theta}^1 & -2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \hat{\theta}^1 \\ \hat{\theta}^2 \\ \hat{\theta}^3 \end{pmatrix} = 0, \quad (2.25)$$

where we rescaled our one-forms $\hat{\theta}$'s as

$$\hat{\theta}^1 \rightarrow \frac{1}{2\Lambda} \hat{\theta}^1, \quad \hat{\theta}^2 \rightarrow \frac{1}{2\Lambda} \hat{\theta}^2 \quad \text{and} \quad \hat{\theta}^3 \rightarrow \frac{1}{2R} \hat{\theta}^3. \quad (2.26)$$

We see that (2.25) defines the Levi-Civita connection with U(3) holonomy group (Kähler structure) on \mathcal{M} if $R = \Lambda$, where R is the radius of S^2 and Λ is the radius of S^4 .

Note that forms $\hat{\theta}^i$ define on \mathcal{M} an integrable almost complex structure \mathcal{J}_+ [1] such that

$$\mathcal{J}_+ \hat{\theta}^i = i \hat{\theta}^i \quad (2.27)$$

with $i = 1, 2, 3$. In other words, $\hat{\theta}^i$'s are (1,0)-forms with respect to (w.r.t.) \mathcal{J}_+ and the manifold \mathcal{M} with such a complex structure can be identified with the Kähler manifold $\mathbb{C}P^3 = \text{SU}(4)/\text{U}(3)$ with the Kähler form

$$\hat{\omega} := \frac{i}{2} \left(\hat{\theta}^1 \wedge \hat{\theta}^{\bar{1}} + \hat{\theta}^2 \wedge \hat{\theta}^{\bar{2}} + \hat{\theta}^3 \wedge \hat{\theta}^{\bar{3}} \right). \quad (2.28)$$

Quasi-Kähler structure on \mathcal{M} . Recall that on the same manifold \mathcal{M} one can introduce the forms

$$\Theta^1 := \hat{\theta}^1, \quad \Theta^2 := \hat{\theta}^2 \quad \text{and} \quad \Theta^3 := \hat{\theta}^{\bar{3}}, \quad (2.29)$$

which are forms of type (1,0) w.r.t. an almost complex structure \mathcal{J}_- [27], $\mathcal{J}_- \Theta^i = i \Theta^i$, which is a never integrable almost complex structure. For Θ^i with the rescaling (2.26) we have

$$d \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} + \begin{pmatrix} -\hat{\alpha}_+ - \alpha_- & \beta_- & 0 \\ -\bar{\beta}_- & -\hat{\alpha}_+ + \alpha_- & 0 \\ 0 & 0 & 2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{pmatrix} = \frac{1}{2R} \begin{pmatrix} \Theta^{\bar{2}} \wedge \Theta^{\bar{3}} \\ \Theta^{\bar{3}} \wedge \Theta^{\bar{1}} \\ \frac{2R^2}{\Lambda^2} \Theta^{\bar{1}} \wedge \Theta^{\bar{2}} \end{pmatrix}. \quad (2.30)$$

The manifold $(\mathcal{M}, \mathcal{J}_-)$ is a quasi-Kähler manifold. Recall that an almost Hermitian $2n$ -manifold with the fundamental (1,1)-form ω is called quasi-Kähler if only (3,0)+(0,3) components of $d\omega$ are non-vanishing [12, 25]. In our case

$$\omega := \frac{i}{2} (\Theta^1 \wedge \Theta^{\bar{1}} + \Theta^2 \wedge \Theta^{\bar{2}} + \Theta^3 \wedge \Theta^{\bar{3}}). \quad (2.31)$$

One can check that for arbitrary ratio Λ/R the (1,2) part of $d\omega$ vanishes and therefore \mathcal{M} is quasi-Kähler [24, 27].

From (2.30) one sees that the manifold $\mathcal{M} = \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ with an almost complex structure \mathcal{J}_- becomes a nearly Kähler manifold if $\Lambda^2 = 2R^2$. Recall that a six-manifold is called nearly Kähler if [12, 24, 25]

$$d\omega = 3\rho \text{Im}\Omega \quad \text{for} \quad \Omega := \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \quad \text{and} \quad d\Omega = 2\rho \omega \wedge \omega, \quad (2.32)$$

where $\rho \in \mathbb{R}$ is proportional to the inverse “radius” $\Lambda = \sqrt{2}R$ of \mathcal{M} .

3 Twistor spaces of \mathbb{CP}^3

Coset representation of \mathbb{CP}^2 . Let us consider the projection

$$\text{SU}(3) \rightarrow \text{SU}(3)/\text{U}(2) = \mathbb{CP}^2. \quad (3.1)$$

One can choose as a coset representative of \mathbb{CP}^2 a local section of the bundle (3.1) given by the matrix

$$V = \frac{1}{\gamma} \begin{pmatrix} 1 & Y^\dagger \\ -Y & W \end{pmatrix} := \frac{1}{\gamma} \begin{pmatrix} 1 & \bar{\lambda}^{\bar{1}} & \bar{\lambda}^{\bar{2}} \\ -\lambda^1 & W_{11} & W_{12} \\ -\lambda^2 & W_{21} & W_{22} \end{pmatrix} \in \text{SU}(3), \quad (3.2)$$

where

$$\gamma^2 := 1 + Y^\dagger Y = 1 + \lambda^1 \bar{\lambda}^{\bar{1}} + \lambda^2 \bar{\lambda}^{\bar{2}} \quad \text{and} \quad W = W^\dagger = \gamma \cdot \mathbf{1}_2 - \frac{1}{\gamma + 1} Y Y^\dagger. \quad (3.3)$$

Here λ^1 and λ^2 are local complex coordinates on a patch of \mathbb{CP}^2 . From (3.2) and (3.3) it is easy to see that

$$WY = Y \quad \text{and} \quad W^2 = \gamma^2 - YY^\dagger \quad \Leftrightarrow \quad V^\dagger V = \mathbf{1}_3 = VV^\dagger. \quad (3.4)$$

Twistor space of $\text{SU}(4)/\text{U}(3)$. Consider the coset space

$$\mathcal{Z} := \text{SU}(4)/\text{U}(2) \times \text{U}(1) \quad (3.5)$$

and the projection

$$\pi : \text{SU}(4)/\text{U}(2) \times \text{U}(1) \rightarrow \text{SU}(4)/\text{U}(3) \cong \mathbb{C}P^3 \quad (3.6)$$

with fibres $\mathbb{C}P^2$. Using the group element (3.2) to parametrize the typical $\mathbb{C}P^2$ -fibre in (3.6), we introduce a flat connection $\tilde{\mathcal{A}}_0$ on the trivial bundle $\mathcal{Z} \times \mathbb{C}^4 \rightarrow \mathcal{Z}$ as

$$\tilde{\mathcal{A}}_0 = \tilde{Q}^{-1} d\tilde{Q} = \tilde{V}^\dagger \hat{\mathcal{A}}_0 \tilde{V} + \tilde{V}^\dagger d\tilde{V} , \quad (3.7)$$

where

$$\tilde{Q} = \hat{Q} \tilde{V} \in \text{SU}(4) \quad \text{and} \quad \tilde{V} := \begin{pmatrix} V & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad V \in \text{SU}(3) . \quad (3.8)$$

The flat connection $\hat{\mathcal{A}}_0$ is given in (2.20) but here we write it as

$$\hat{\mathcal{A}}_0 = \begin{pmatrix} \alpha_- & -\bar{\beta}_- & -\hat{\theta}^2 & -\hat{\theta}^{\bar{1}} \\ \beta_- & -\alpha_- & \hat{\theta}^1 & -\hat{\theta}^{\bar{2}} \\ \hat{\theta}^{\bar{2}} & -\hat{\theta}^{\bar{1}} & \hat{\alpha}_+ & -\hat{\theta}^{\bar{3}} \\ \hat{\theta}^1 & \hat{\theta}^2 & \hat{\theta}^3 & -\hat{\alpha}_+ \end{pmatrix} =: \begin{pmatrix} B & -T \\ T^\dagger & -\hat{\alpha}_+ \end{pmatrix} , \quad (3.9)$$

where

$$B = \begin{pmatrix} \alpha_- & -\bar{\beta}_- & -\hat{\theta}^2 \\ \beta_- & -\alpha_- & \hat{\theta}^1 \\ \hat{\theta}^{\bar{2}} & -\hat{\theta}^{\bar{1}} & \hat{\alpha}_+ \end{pmatrix} , \quad T := \begin{pmatrix} \hat{\theta}^{\bar{1}} \\ \hat{\theta}^{\bar{2}} \\ \hat{\theta}^{\bar{3}} \end{pmatrix} \quad \text{and} \quad T^\dagger = (\hat{\theta}^1 \ \hat{\theta}^2 \ \hat{\theta}^3) . \quad (3.10)$$

Using (3.7), we obtain the connection

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} V^\dagger B V + V^\dagger dV & -V^\dagger T \\ T^\dagger V & -\hat{\alpha}_+ \end{pmatrix} =: \begin{pmatrix} \tilde{B} & -\tilde{T} \\ \tilde{T}^\dagger & -\hat{\alpha}_+ \end{pmatrix} \quad \text{with} \quad \tilde{T} = \begin{pmatrix} \tilde{\theta}^{\bar{1}} \\ \tilde{\theta}^{\bar{2}} \\ \tilde{\theta}^{\bar{3}} \end{pmatrix} \quad (3.11)$$

and for the curvature $\tilde{\mathcal{F}}_0 = d\tilde{\mathcal{A}}_0 + \tilde{\mathcal{A}}_0 \wedge \tilde{\mathcal{A}}_0$ we get

$$\tilde{\mathcal{F}}_0 = \begin{pmatrix} d\tilde{B} + \tilde{B} \wedge \tilde{B} - \tilde{T} \wedge \tilde{T}^\dagger & -d\tilde{T} - (\tilde{B} + \hat{\alpha}_+ \cdot \mathbf{1}_3) \wedge \tilde{T} \\ d\tilde{T}^\dagger + \tilde{T}^\dagger \wedge (\tilde{B} + \hat{\alpha}_+ \cdot \mathbf{1}_3) & -d\hat{\alpha}_+ - \tilde{T}^\dagger \wedge \tilde{T} \end{pmatrix} . \quad (3.12)$$

We have

$$\tilde{B} = V^\dagger B V + V^\dagger dV =: \begin{pmatrix} \tilde{\alpha}_- & \Upsilon^\dagger \\ -\Upsilon & \Sigma \end{pmatrix} \quad (3.13)$$

with

$$\Sigma =: \begin{pmatrix} \tilde{a} - \tilde{\alpha}_- & -\bar{b} \\ b & -\tilde{a} + \hat{\alpha}_+ \end{pmatrix} \quad \text{and} \quad \Upsilon =: \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} . \quad (3.14)$$

Flatness $\tilde{\mathcal{F}}_0 = 0$ of the connection (3.11) yields

$$d \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} + \begin{pmatrix} -\tilde{\alpha}_- - \hat{\alpha}_+ & 0 & 0 \\ 0 & -\tilde{a} + \tilde{\alpha}_- - \hat{\alpha}_+ & -b \\ 0 & \bar{b} & \tilde{a} - 2\hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{24} + \tilde{\theta}^{35} \\ -\tilde{\theta}^{14} \\ -\tilde{\theta}^{15} \end{pmatrix} . \quad (3.15)$$

From

$$d\tilde{B} + \tilde{B} \wedge \tilde{B} - \tilde{T} \wedge \tilde{T}^\dagger = 0 \quad (3.16)$$

it follows that

$$d \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} + \begin{pmatrix} \tilde{a} - 2\tilde{\alpha}_- & -\bar{b} \\ b & -\tilde{a} + \hat{\alpha}_+ - \tilde{\alpha}_- \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{1\bar{2}} \\ \tilde{\theta}^{1\bar{3}} \end{pmatrix}. \quad (3.17)$$

We obtain

$$d \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \\ \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} + \begin{pmatrix} -\tilde{\alpha}_- - \hat{\alpha}_+ & 0 & 0 & 0 & 0 \\ 0 & -\tilde{a} + \tilde{\alpha}_- - \hat{\alpha}_+ & -b & 0 & 0 \\ 0 & \bar{b} & \tilde{a} - 2\hat{\alpha}_+ & 0 & 0 \\ 0 & 0 & 0 & \tilde{a} - 2\tilde{\alpha}_- & -\bar{b} \\ 0 & 0 & 0 & b & -\tilde{a} - \tilde{\alpha}_- + \hat{\alpha}_+ \end{pmatrix} \wedge \begin{pmatrix} \tilde{\theta}^1 \\ \tilde{\theta}^2 \\ \tilde{\theta}^3 \\ \tilde{\theta}^4 \\ \tilde{\theta}^5 \end{pmatrix} = \begin{pmatrix} \tilde{\theta}^{24} + \frac{\Lambda}{R} \tilde{\theta}^{35} \\ -\tilde{\theta}^{1\bar{4}} \\ -\frac{R}{\Lambda} \tilde{\theta}^{1\bar{5}} \\ \frac{1}{4\Lambda^2} \tilde{\theta}^{1\bar{2}} \\ \frac{1}{4\Lambda R} \tilde{\theta}^{1\bar{3}} \end{pmatrix}, \quad (3.18)$$

where we rescaled our $\tilde{\theta}^a$ with $a = 1, \dots, 5$ as in (2.26):

$$\tilde{\theta}^1 \rightarrow \frac{1}{2\Lambda} \tilde{\theta}^1, \quad \tilde{\theta}^2 \rightarrow \frac{1}{2\Lambda} \tilde{\theta}^2, \quad \tilde{\theta}^3 \rightarrow \frac{1}{2R} \tilde{\theta}^3, \quad \tilde{\theta}^4 \rightarrow \tilde{\theta}^4 \quad \text{and} \quad \tilde{\theta}^5 \rightarrow \tilde{\theta}^5. \quad (3.19)$$

The manifold $\text{SU}(4)/\text{U}(2) \times \text{U}(1)$ is the twistor space for the Kähler space $\mathbb{C}P^3 = \text{SU}(4)/\text{U}(3)$ for $\Lambda^2 = R^2$. Forms $\tilde{\theta}^a$ define on $\text{SU}(4)/\text{U}(2) \times \text{U}(1)$ an integrable almost complex structure $\tilde{\mathcal{J}}_+$ such that

$$\tilde{\mathcal{J}}_+ \tilde{\theta}^a = i \tilde{\theta}^a. \quad (3.20)$$

In the Kähler case we choose $\Lambda = R = \frac{1}{2}$.

Twistor space of $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$. Consider the coset space

$$\mathcal{Z}' := \text{Sp}(2)/\text{U}(1) \times \text{U}(1) \quad (3.21)$$

and the projection

$$\pi' : \text{Sp}(2)/\text{U}(1) \times \text{U}(1) \rightarrow \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \cong \mathbb{C}P^3 \quad (3.22)$$

with fibres $\mathbb{C}P^1 \cong \text{Sp}(1)/\text{U}(1)$. We choose the group element

$$\hat{g} = \frac{1}{(1 + \lambda\bar{\lambda})^{\frac{1}{2}}} \begin{pmatrix} 1 & -\bar{\lambda} \\ \lambda & 1 \end{pmatrix} \in \text{SU}(2) \cong \text{Sp}(1) \quad (3.23)$$

to parametrize the typical $\mathbb{C}P^1$ -fibre in (3.22), where λ is a local complex coordinate on the Riemann sphere $\mathbb{C}P^1$. By formula

$$\hat{g}^{-1} d\hat{g} =: \begin{pmatrix} \hat{a} & -\theta^{\bar{4}} \\ \theta^4 & -\hat{a} \end{pmatrix} \quad (3.24)$$

where

$$\hat{a} := \frac{1}{2(1 + \lambda\bar{\lambda})} (\bar{\lambda} d\lambda - \lambda d\bar{\lambda}), \quad (3.25)$$

we introduce on $\mathbb{C}P^1$ the forms

$$\theta^4 = \frac{d\lambda}{1 + \lambda\bar{\lambda}} \quad \text{and} \quad \theta^{\bar{4}} = \frac{d\bar{\lambda}}{1 + \lambda\bar{\lambda}} \quad (3.26)$$

of type (1,0) and (0,1), respectively.

Using the group element (3.23), we introduce a flat connection \mathcal{A}'_0 on the trivial bundle $\mathcal{Z}' \times \mathbb{C}^4 \rightarrow \mathcal{Z}'$ as

$$\mathcal{A}'_0 = \check{Q}^{-1} d\check{Q} = \hat{G}^\dagger \hat{A}_0 \hat{G} + \hat{G}^\dagger d\hat{G} , \quad (3.27)$$

where

$$\check{Q} = \hat{Q} \hat{G} \in \mathrm{Sp}(2) \quad \text{and} \quad \hat{G} := \begin{pmatrix} \hat{g} & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} \in \mathrm{Sp}(1) \subset \mathrm{Sp}(2) . \quad (3.28)$$

The flat connection \hat{A}_0 is given in (2.20) and (3.9). Using (3.27), we obtain the connection

$$\mathcal{A}'_0 = \begin{pmatrix} \hat{g}^\dagger \hat{A}^- \hat{g} + \hat{g}^\dagger d\hat{g} & -\hat{g}^\dagger \hat{\phi} \\ \hat{\phi}^\dagger \hat{g} & \hat{A}^+ \end{pmatrix} =: \begin{pmatrix} \check{A}^- & -\check{\phi} \\ \check{\phi}^\dagger & \check{A}^+ \end{pmatrix} \quad (3.29)$$

with

$$\check{\phi} = \hat{g}^\dagger \hat{\phi} = \frac{1}{(1 + \lambda \bar{\lambda})^{1/2}} \begin{pmatrix} \hat{\theta}^2 - \bar{\lambda} \hat{\theta}^1 & \hat{\theta}^{\bar{1}} + \bar{\lambda} \hat{\theta}^{\bar{2}} \\ -\hat{\theta}^1 - \lambda \hat{\theta}^2 & \hat{\theta}^{\bar{2}} - \lambda \hat{\theta}^{\bar{1}} \end{pmatrix} =: \begin{pmatrix} \check{\theta}^2 & \check{\theta}^{\bar{1}} \\ -\check{\theta}^1 & \check{\theta}^{\bar{2}} \end{pmatrix} , \quad (3.30)$$

$$\check{A}^+ := \begin{pmatrix} \check{\alpha}_+ & -\check{\theta}^{\bar{3}} \\ \check{\theta}^3 & -\check{\alpha}_+ \end{pmatrix} = \begin{pmatrix} \hat{\alpha}_+ & -\hat{\theta}^{\bar{3}} \\ \hat{\theta}^3 & -\hat{\alpha}_+ \end{pmatrix} = \hat{A}^+ \quad \text{and} \quad \check{A}^- := \begin{pmatrix} \check{\alpha}_- & -\check{\theta}^{\bar{4}} \\ \check{\theta}^4 & -\check{\alpha}_- \end{pmatrix} , \quad (3.31)$$

where

$$\check{\alpha}_- = \frac{1}{1 + \lambda \bar{\lambda}} \{ (1 - \lambda \bar{\lambda}) \alpha_- + \bar{\lambda} \beta_- - \lambda \bar{\beta}_- + \frac{1}{2} (\bar{\lambda} d\lambda - \lambda d\bar{\lambda}) \} , \quad (3.32)$$

$$\check{\theta}^4 = \frac{1}{1 + \lambda \bar{\lambda}} \{ d\lambda + \beta_- - 2\lambda \alpha_- + \lambda^2 \bar{\beta}_- \} , \quad \check{\theta}^{\bar{4}} := \overline{\check{\theta}^4} . \quad (3.33)$$

For the curvature $\mathcal{F}'_0 = d\mathcal{A}'_0 + \mathcal{A}'_0 \wedge \mathcal{A}'_0$ we get

$$\mathcal{F}'_0 = \begin{pmatrix} d\check{A}^- + \check{A}^- \wedge \check{A}^- - \check{\phi} \wedge \check{\phi}^\dagger & -d\check{\phi} - \check{A}^- \wedge \check{\phi} - \check{\phi} \wedge \check{A}^+ \\ d\check{\phi}^\dagger + \check{\phi}^\dagger \wedge \check{A}^- + \check{A}^+ \wedge \check{\phi}^\dagger & d\check{A}^+ + \check{A}^+ \wedge \check{A}^+ - \check{\phi}^\dagger \wedge \check{\phi} \end{pmatrix} . \quad (3.34)$$

From the flatness $\mathcal{F}'_0 = 0$ of the connection (3.29) we obtain the Maurer-Cartan equations

$$d \begin{pmatrix} \check{\theta}^1 \\ \check{\theta}^2 \\ \check{\theta}^3 \\ \check{\theta}^4 \end{pmatrix} + \begin{pmatrix} -\check{\alpha}_- - \check{\alpha}_+ & 0 & 0 & 0 \\ 0 & \check{\alpha}_- - \check{\alpha}_+ & 0 & 0 \\ 0 & 0 & -2\check{\alpha}_+ & 0 \\ 0 & 0 & 0 & -2\check{\alpha}_- \end{pmatrix} \wedge \begin{pmatrix} \check{\theta}^1 \\ \check{\theta}^2 \\ \check{\theta}^3 \\ \check{\theta}^4 \end{pmatrix} = \begin{pmatrix} -\check{\theta}^{24} - \check{\theta}^{3\bar{2}} \\ \check{\theta}^{3\bar{1}} + \check{\theta}^{1\bar{4}} \\ 2\check{\theta}^{12} \\ -2\check{\theta}^{1\bar{2}} \end{pmatrix} , \quad (3.35)$$

which define the $u(1) \oplus u(1)$ torsionful connection on the twistor space $\mathcal{Z}' = \mathrm{Sp}(2)/\mathrm{U}(1) \times \mathrm{U}(1)$. Forms $\check{\theta}^a$ in (3.35) with $a = 1, \dots, 4$ define on \mathcal{Z}' an integrable almost complex structure I'_+ such that

$$I'_+ \check{\theta}^a = i \check{\theta}^a . \quad (3.36)$$

Its integrability follows from the vanishing (0,2)-type components of the torsion on the right hand side of (3.35).

4 Twistor description of instanton bundles over \mathbb{CP}^3

Instanton bundles over \mathbb{CP}^3 . Consider a complex vector bundle \mathcal{E} over \mathbb{CP}^3 with a connection one-form \mathcal{A} having the curvature \mathcal{F} . Recall that $(\mathcal{E}, \mathcal{A})$ is called an instanton bundle if \mathcal{A} satisfies the Hermitian Yang-Mills (HYM) equations,¹ which on \mathbb{CP}^3 can be written in the form

$$\mathcal{F}^{0,2} = 0 = \mathcal{F}^{2,0} \quad \Leftrightarrow \quad \hat{\Omega} \wedge \mathcal{F} = 0, \quad (4.1)$$

$$\hat{\omega} \lrcorner \mathcal{F} = 0 \quad \Leftrightarrow \quad \hat{\omega} \wedge \hat{\omega} \wedge \mathcal{F} = 0, \quad (4.2)$$

where the notation $\hat{\omega} \lrcorner$ exploits the underlying Riemannian metric $g = \delta_{\hat{a}\hat{b}} e^{\hat{a}} e^{\hat{b}}$ on \mathbb{CP}^3 , $\hat{a}, \hat{b}, \dots = 1, \dots, 6$. Here, $\hat{\omega}$ given in (2.28) is a (1,1)-form, and $\hat{\Omega} := \hat{\theta}^1 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3$ is a locally defined (3,0)-form on \mathbb{CP}^3 . Recall that, from the point of view of algebraic geometry, (4.1) means that the bundle $\mathcal{E} \rightarrow \mathbb{CP}^3$ is holomorphic and (4.2) means that \mathcal{E} is a polystable vector bundle [17]. In fact, in the right hand side of (4.2) one can add the term $\beta \hat{\omega} \wedge \hat{\omega} \wedge \mathcal{F}$ with β proportional to the first Chern number $c_1(\mathcal{E})$, but we assume $c_1(\mathcal{E}) = 0$ since for a bundle with field strength \mathcal{F} of non-zero degree one can obtain a degree-zero bundle by considering $\tilde{\mathcal{F}} = \mathcal{F} - \frac{1}{r} (\text{tr } \mathcal{F}) \cdot \mathbf{1}_r$, where $r = \text{rank } \mathcal{E}$.

Pull-back to \mathcal{Z} . Consider the twistor fibration (3.6). Let $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = (\pi^* \mathcal{E}, \pi^* \mathcal{A})$ be the pulled-back instanton bundle over \mathcal{Z} with the curvature $\tilde{\mathcal{F}} = d\tilde{\mathcal{A}} + \tilde{\mathcal{A}} \wedge \tilde{\mathcal{A}}$. We have

$$\tilde{\mathcal{F}} = \frac{1}{2} \tilde{\mathcal{F}}_{ab} \tilde{\theta}^a \wedge \tilde{\theta}^b + \tilde{\mathcal{F}}_{a\bar{b}} \tilde{\theta}^a \wedge \tilde{\theta}^{\bar{b}} + \frac{1}{2} \tilde{\mathcal{F}}_{\bar{a}\bar{b}} \tilde{\theta}^{\bar{a}} \wedge \tilde{\theta}^{\bar{b}} = \pi^* \mathcal{F} \quad (4.3)$$

with $a, b, \dots = 1, \dots, 5$. Using the relation between $\tilde{\theta}^a$ and $\hat{\theta}^a$ described in Section 3, we obtain

$$\tilde{\mathcal{F}}_{i\bar{j}} = C_i^{\bar{k}} C_{\bar{j}}^{\bar{l}} \mathcal{F}_{\bar{k}\bar{l}} \quad \text{and} \quad \tilde{\mathcal{F}}_{i\bar{j}} = \bar{C}_i^k C_{\bar{j}}^{\bar{l}} \mathcal{F}_{k\bar{l}}, \quad (4.4)$$

where $C = \bar{V}^\dagger$ with

$$\begin{aligned} C_1^{\bar{1}} &= \frac{1}{\gamma}, \quad C_2^{\bar{1}} = -\frac{\lambda^1}{\gamma}, \quad C_3^{\bar{1}} = -\frac{\lambda^2}{\gamma}, \\ C_1^{\bar{2}} &= \frac{\bar{\lambda}^1}{\gamma}, \quad C_2^{\bar{2}} = \frac{\gamma + 1 + \lambda^2 \bar{\lambda}^2}{\gamma(\gamma + 1)}, \quad C_3^{\bar{2}} = -\frac{\lambda^2 \bar{\lambda}^1}{\gamma(\gamma + 1)}, \\ C_1^{\bar{3}} &= \frac{\bar{\lambda}^2}{\gamma}, \quad C_2^{\bar{3}} = -\frac{\lambda^1 \bar{\lambda}^2}{\gamma(\gamma + 1)}, \quad C_3^{\bar{3}} = \frac{\gamma + 1 + \lambda^1 \bar{\lambda}^1}{\gamma(\gamma + 1)}, \end{aligned} \quad (4.5)$$

and \bar{C} is the complex conjugate matrix. Thus, more explicitly, we get

$$\tilde{\mathcal{F}}_{1\bar{2}} = \frac{1}{\gamma} \left\{ \frac{\gamma + 1 + \lambda^1 \bar{\lambda}^1}{\gamma + 1} \mathcal{F}_{1\bar{2}} - \frac{\lambda^1 \bar{\lambda}^2}{\gamma + 1} \mathcal{F}_{3\bar{1}} - \bar{\lambda}^2 \mathcal{F}_{2\bar{3}} \right\}, \quad (4.6)$$

$$\tilde{\mathcal{F}}_{3\bar{1}} = \frac{1}{\gamma} \left\{ \frac{\gamma + 1 + \lambda^2 \bar{\lambda}^2}{\gamma + 1} \mathcal{F}_{3\bar{1}} - \frac{\lambda^2 \bar{\lambda}^1}{\gamma + 1} \mathcal{F}_{1\bar{2}} - \bar{\lambda}^1 \mathcal{F}_{2\bar{3}} \right\}, \quad (4.7)$$

$$\tilde{\mathcal{F}}_{2\bar{3}} = \frac{1}{\gamma} \left\{ \mathcal{F}_{2\bar{3}} + \lambda^1 \mathcal{F}_{3\bar{1}} + \lambda^2 \mathcal{F}_{1\bar{2}} \right\}, \quad (4.8)$$

¹These equations are also called the Donaldson-Uhlenbeck-Yau equations.

$$\tilde{\mathcal{F}}_{i\bar{4}} = \tilde{\mathcal{F}}_{i\bar{5}} = 0, \quad (4.9)$$

$$\tilde{\mathcal{F}}_{1\bar{1}} + \tilde{\mathcal{F}}_{2\bar{2}} + \tilde{\mathcal{F}}_{3\bar{3}} + \tilde{\mathcal{F}}_{4\bar{4}} + \tilde{\mathcal{F}}_{5\bar{5}} = \mathcal{F}_{1\bar{1}} + \mathcal{F}_{2\bar{2}} + \mathcal{F}_{3\bar{3}}. \quad (4.10)$$

The vanishing of $\tilde{\mathcal{F}}_{2\bar{3}}$ for all values of $(\lambda^1, \lambda^2) \in \mathbb{C}P^2$ is equivalent to the holomorphicity equation (4.1). In homogeneous coordinates y^i on $\mathbb{C}P^2$ ($\lambda^1 = y^2/y^1$, $\lambda^2 = y^3/y^1$, $y^1 \neq 0$), this condition can be written as

$$\tilde{\mathcal{F}}_{2\bar{3}} = 0 \quad \Leftrightarrow \quad y^i \varepsilon_{ijk} \mathcal{F}^{jk} = 0, \quad (4.11)$$

where the indices \bar{i}, \bar{j}, \dots are raised with the metric $\delta^{i\bar{j}}$. From (4.6)-(4.9) we see that the bundle $\tilde{\mathcal{E}}$ is holomorphic for holomorphic \mathcal{E} as well as polystable due to (4.2), (4.10) and it is holomorphically trivial after restricting to the fibres $\mathbb{C}P_x^2 \hookrightarrow \mathcal{Z}$ of the projection π for each $x \in \mathbb{C}P^3$.

Pull-back to \mathcal{Z}' . Consider now the twistor fibration (3.22) and the pulled-back instanton bundle $(\mathcal{E}', \mathcal{A}') = (\pi'^*\mathcal{E}, \pi'^*\mathcal{A})$ over \mathcal{Z}' with the curvature $\mathcal{F}' = d\mathcal{A}' + \mathcal{A}' \wedge \mathcal{A}'$. We again have the relation (4.3) with $a, b, \dots = 1, \dots, 4$. For the matrix C in (4.4) we now find

$$C = \begin{pmatrix} \varkappa & \varkappa\lambda & 0 \\ -\varkappa\bar{\lambda} & \varkappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad \varkappa = (1 + \lambda\bar{\lambda})^{-\frac{1}{2}}, \quad (4.12)$$

where λ is a local complex coordinate on $\mathbb{C}P^1$ used in (3.23)-(3.26).

Using (4.12), we obtain

$$\mathcal{F}'_{1\bar{2}} = \mathcal{F}_{1\bar{2}}, \quad \mathcal{F}'_{3\bar{1}} = \varkappa(\mathcal{F}_{3\bar{1}} + \bar{\lambda}\mathcal{F}_{2\bar{3}}), \quad \mathcal{F}'_{2\bar{3}} = \varkappa(\mathcal{F}_{2\bar{3}} - \lambda\mathcal{F}_{3\bar{1}}), \quad \mathcal{F}'_{i\bar{4}} = 0, \quad (4.13)$$

$$\mathcal{F}'_{1\bar{1}} + \mathcal{F}'_{2\bar{2}} + \mathcal{F}'_{3\bar{3}} + \mathcal{F}'_{4\bar{4}} = \mathcal{F}_{1\bar{1}} + \mathcal{F}_{2\bar{2}} + \mathcal{F}_{3\bar{3}}. \quad (4.14)$$

Therefore, instanton bundles $(\mathcal{E}, \mathcal{A})$ over the nonsymmetric Kähler coset space $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \cong \mathbb{C}P^3$ are pulled back to holomorphic polystable bundles $(\mathcal{E}', \mathcal{A}')$ over the complex twistor space $\mathcal{Z}' = \text{Sp}(2)/\text{U}(1) \times \text{U}(1)$. Furthermore, \mathcal{E}' is flat along the fibres $\mathbb{C}P_x^1$ of the bundle (3.22), and one can set the components of \mathcal{A}' along the fibres equal to zero. Thus, restrictions of the vector bundle \mathcal{E}' to fibres $\mathbb{C}P_x^1 \hookrightarrow \mathcal{Z}'$ of the projection π' are holomorphically trivial for each $x \in \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1) \cong \mathbb{C}P^3$. Note that (4.13) and (4.14) can be obtained from (4.6)-(4.10) by putting $\lambda^1 = -\lambda$ and $\lambda^2 = 0$. Then (3.11) will coincide with (3.29) after the substitution $\tilde{\theta}^4 \rightarrow -\tilde{\theta}^4$, $\tilde{\theta}^5 \rightarrow -\tilde{\theta}^2$, $b \rightarrow -\tilde{\theta}^1$ etc. This correspondence follows from the fact that \mathcal{Z}' is a complex (codimension one) submanifold of the twistor space \mathcal{Z} .

We have seen that solutions of the Hermitian Yang-Mills equations on the complex projective space $\mathbb{C}P^3$, represented either as the symmetric space $\text{SU}(4)/\text{U}(3)$ or as the nonsymmetric homogeneous space $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$, have a twistor description similar to the four-dimensional case but valid for the reduced twistor spaces \mathcal{Z} and \mathcal{Z}' with fibres $\mathbb{C}P^2$ and $\mathbb{C}P^1$, respectively, instead of the metric twistor space $\text{Tw}(\mathbb{C}P^3) = P(\mathbb{C}P^3, \text{SO}(6)) \times_{\text{SO}(6)} \text{SO}(6)/\text{U}(3) \cong \mathbb{C}P^3$.

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